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AN ANALYTICAL SOLUTION OF THE PROBLEM OF CONVECTIVE DIFFUSION IN THE NEIGHBOURHOOD OF A DISCONTINUITY OF THE CATALYTIC PROPERTIES OF A SURFACE*

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The problem of convective diffusion when a binary mixture flows round a plate when there is a line of discontinuity of the catalytic properties on the plate is considered. The effect of longitudinal diffusion is taken into account. The surface is assumed to be non-catalytic up to the discontinuity but possesses a finite catalytic activity after the discontinuity. At low values of the coefficient of catalytic activity, an analytic solution of the problem is obtained by the application of a Fourier transform. The asymptotic forms of the solution are found in the form of simple formulae both near the remote from the point of discontinuity of the boundary conditions and both upstream and downstream. A comparison is made with the solutions obtained in the boundary layer approximation and by a numerical method /1/.

The problem of convective diffusion (or thermal conductivity) in the case of a transition from a not-catalytic surface onto an ideally catalytic surface has been solved /2, 3/ by the Wiener-Hopf method.

1. The stationary flow of a two component incompressible liquid or gas with constant diffusion properties and a linear velocity profile $(u' = VL^{-1}y',\ v' = 0)$ in the x' direction around an infinite plate $y' = 0$, on the surface of which a heterogeneous first-order reaction occurs, is considered. The surface is assumed to be non-catalytic in the half-plane $y' = 0, x' < 0$ and to possess a finite catalytic activity in the half plane $y' = 0, x' > 0$.

The diffusion equation (which is identical in form to the heat conduction equation) and the boundary conditions in this case have the form

$$y\frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}, \quad -\infty < x < \infty, \quad y > 0 \tag{1.1}$$

$$x \to -\infty,\ \forall y \ \text{ and } y \to \infty,\ \forall x\colon c \to c_0 \tag{1.2}$$

$$y = 0,\quad x < 0\colon\ \frac{\partial c}{\partial y} = 0;\quad y = 0,\quad x > 0\colon\ \frac{\partial c}{\partial y} = kc;$$

$$k = \left(\frac{L}{VD}\right)^{1/3} k'$$

Here k' is the rate constant for the heterogeneous recombination on the surface, and the dimensionless variables x and y are related to the dimensional variables in the following manner:

$$x' = \left(\frac{DL}{V}\right)^{1/3} x,\quad y' = \left(\frac{DL}{V}\right)^{1/3} y$$

(D is the coefficient of diffusion and c is the concentration).

Let us consider the case when $k \ll 1$. A solution of the problem can then be sought in the form

$$c = c_0(1 - kf + \ldots) \tag{1.3}$$

For the function $f(x, y)$ we obtain an equation, which is identical to (1.1), while the boundary conditions take the form

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$$x \to -\infty,\ \forall y \quad \text{and} \quad y \to \infty,\ \forall x: f \to 0$$
$$y=0,\quad x<0:\ \frac{\partial f}{\partial y}=0;\quad y=0,\ x>0:\ \frac{\partial f}{\partial y}=-1 \tag{1.4}$$

The problem is elliptic and the system of boundary conditions (1.4) must therefore be augmented with the conditions when $x \to \infty$, $\forall y$.

We will seek a bounded solution of the problem $f \to 0$ as $x \to \infty$, $\forall y$. In doing this, instead of the last boundary condition of (1.4), let us consider the condition

$$y = 0, x > 0: \partial f/\partial y = -e^{-\delta x}, \delta > 0 \tag{1.5}$$

A certain positive parameter δ is introduced which may be made as small as desired (the introduction of this parameter will allow us to avoid the singularity which arises in passing around the branching point in finding the function $G_+(\alpha)$). After finding the solution in the neighbourhood of the discontinuity, we allow δ to tend to zero and find the solution which corresponds to the last boundary condition of (1.4).

The mathematical formulation of the problem is therefore as follows: it is required that a function $f(x, y)$ be found which satisfies the equation and the boundary conditions:

$$y\frac{\partial f}{\partial x}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} \tag{1.6}$$

$$(x^2+y^2)^{1/2} \to \infty:\ f \to 0 \tag{1.7}$$
$$y=0,\quad x<0:\ \frac{\partial f}{\partial y}=0;\quad y=0,\quad x>0:\ \frac{\partial f}{\partial y}=-e^{-\delta x},\quad \delta>0$$

Let us make one further assumption concerning the behaviour of the solution as $x \to -\infty$:

$$|f| \leqslant Ae^{ax} \tag{1.8}$$

for a certain $a > 0$. This assumption will subsequently be checked after the solution of the problem has been found.

2. Let us introduce the Fourier transform with respect to x

$$\Phi(\alpha, y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, y)\, e^{i\alpha x}\, dx,\quad \alpha=\sigma+i\tau \tag{2.1}$$

It defines the function $\Phi(\alpha, y)$ which is analytic with respect to the variable α in the strip $0 < \operatorname{Im}\alpha < a$ by virtue of condition (1.8) and the boundedness of f as $x \to +\infty$ and, moreover, the inverse transform has the form

$$f(x, y)=\frac{1}{\sqrt{2\pi}}\int_{i\tau'-\infty}^{i\tau'+\infty}\Phi(\alpha, y)\, e^{-i\alpha x}\, d\alpha,\quad 0<\tau'<a \tag{2.2}$$

By making the substitution

$$s = e^{-i\pi/3}(i\alpha)^{1/3}(y+i\alpha) \tag{2.3}$$

for the function $\Phi(\alpha, s)$ from (1.6), we obtain Airy's equation

$$d^2\Phi/ds^2 - s\Phi = 0 \tag{2.4}$$

It is necessary to choose a solution of this equation which decreases $(\Phi \to 0)$ as $y \to \infty$. This solution is expressed in terms of the Airy functions $\operatorname{Ai}(s)$ /4/ (we assume that $\alpha^{1/3} = \sigma^{1/3}$ on the real positive axis)

$$\Phi(\alpha, y)=A(\alpha)\operatorname{Ai}(s),\quad \operatorname{Ai}(s)=\frac{1}{\pi}\int_0^{\infty}\cos\left(ts+\frac{t^3}{3}\right)dt \tag{2.5}$$

and $-\pi/3 < \arg s < \pi/3$ (in this interval, the Airy function decays exponentially as $s \to \infty$). It follows from this that

$$-\pi/2 < \arg\alpha < 3\pi/2$$

this is, the α plane must have a cut along the negative imaginary axis.

In order to determine $A(\alpha)$, let us differentiate (2.5) with respect to y and write the resulting relationship when $y = 0$

$$\frac{\partial\Phi}{\partial y}(\alpha, 0)=A(\alpha)\operatorname{Ai}'(z)\, e^{-i\pi/6}\alpha^{1/3},\quad z=s\,|_{y=0}=e^{i\pi/3}\alpha^{4/3} \tag{2.6}$$

By taking account of the boundary conditions (1.7), we obtain an expression for the left-hand side of the first equality in (2.6), after which, we find

$$A(\alpha) = e^{i\pi/6}\,[\sqrt{2\pi}\,(i\alpha - \delta)\alpha^{1/3}\mathrm{Ai}'(z)]^{-1} \tag{2.7}$$

By substituting this expression into (2.5), we determine $\Phi(\alpha, y)$ and can then find $f(x, y)$ from (2.2).

Let us split the function $\Phi(\alpha, y)$ up into two functions:

$$\Phi(\alpha, y) = \Phi_+(\alpha, y) + \Phi_-(\alpha, y) \tag{2.8}$$

$$\Phi_\pm(\alpha, y) = \pm\frac{1}{\sqrt{2\pi}}\int_0^{\pm\infty} f(x, y)\, e^{i\alpha x}\, dx \tag{2.9}$$

We shall now determine the functions $\Phi_+(\alpha, 0)$ and $\Phi_-(\alpha, 0)$ by means of the Wiener-Hopf method which makes it possible to find the asymptotic forms of the behaviour of the solution as $x \to \pm 0$ and $x \to \pm\infty$. Let us write (2.8) when $y = 0$

$$\Phi_+(\alpha, 0) + \Phi_-(\alpha, 0) = G(\alpha) \tag{2.10}$$

$$G(\alpha) = \frac{e^{i\pi/2} g(z)}{\sqrt{2\pi}\,\alpha(\alpha + i\delta)}, \quad g(z) = \frac{K_{1/3}(^2/_3 z^{3/2})}{K_{2/3}(^2/_3 z^{3/2})} \tag{2.11}$$

(here the fact that the Airy function can be expressed in terms of modified Bessel functions of the second kind with an order of $^1/_3$ /4/ has been taken account of: $\mathrm{Ai}(z) = \pi^{-1}(z/3)^{1/2} K_{1/3}(^2/_3 z^{3/2})$ as well as the rule for the differentiation of Bessel functions /5/).

It is required that we should determine the functions $\Phi_+(\alpha, 0)$ and $\Phi_-(\alpha, 0)$ from the functional Eq.(2.10). These functions are analytic in the $\mathrm{Im}\,\alpha > 0$ and $\mathrm{Im}\,\alpha < a$ half planes respectively (by virtue of (1.7) and (1.8)) and tend to zero as $|\alpha| \to \infty$ in both regions where they are analytic.

Suppose it is possible to represent the function $G(\alpha)$ in the form

$$G(\alpha) = G_+(\alpha) + G_-(\alpha) \tag{2.12}$$

where the functions $G_+(\alpha)$ and $G_-(\alpha)$ are analytic in the half planes $\mathrm{Im}\,\alpha > 0$ and $\mathrm{Im}\,\alpha < a$ respectively. Then, by applying the reasoning of the Wiener-Hopf method /6/, we obtain that

$$\Phi_+(\alpha, 0) = G_+(\alpha), \quad \Phi_-(\alpha, 0) = G_-(\alpha) \tag{2.13}$$

(In order that (2.13) should be satisfied, it is necessary that $G_-(\alpha) \to 0$ and $G_+(\alpha) \to 0$ as $|\alpha| \to \infty$. This will be proved later after the actual form of G_+ and G_- has been found).

So, in order to determine the functions Φ_+ and Φ_-, it is necessary to find a representation of $G(\alpha)$ in the form of (2.12).

3. Let us determine the functions G_+ and G_-. The functions $K_{1/3}$ and $K_{2/3}$ in the α plane have an infinite number of simple zeros on the positive imaginary axis and a branch point $\alpha = 0$ (the α plane is cut along the negative imaginary axis). Let us now choose the parameter a in (1.9) to be smaller in magnitude than the distance from the origin of coordinates up to the first zero of both the function $K_{1/3}$ and the function $K_{2/3}$. The function $G(\alpha)$ will then by analytic in the domain $0 < \mathrm{Im}\,\alpha < a$ ($G(\alpha)$ has an infinite number of simple poles on the positive imaginary axis, a simple pole $\alpha = -i\delta$ on the negative imaginary axis and a branch point $\alpha = 0$). Furthermore, $|G(\alpha)| = O(|\alpha|^{-2})$, $|\alpha| \to \infty$. Consequently /6/, the function $G(\alpha)$ can be represented in the form of a sum

$$G(\alpha) = G_+(\alpha) + G_-(\alpha) \tag{3.1}$$

$$G_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic'}^{\alpha+ic'} \frac{G(\zeta)}{\zeta - \alpha}\, d\zeta, \quad G_-(\alpha) = -\frac{1}{2\pi i}\int_{-\alpha+id}^{\infty+id} \frac{G(\zeta)}{\zeta - \alpha}\, d\zeta$$

$$0 < c' < \mathrm{Im}\,\alpha < d < a$$

where the functions $G_+(\alpha)$ and $G_-(\alpha)$ are analytic in the upper ($\mathrm{Im}\,\alpha > 0$) and lower ($\mathrm{Im}\,\alpha < a$) half planes respectively.

Let us now consider the function $G_+(\alpha)$ and deform the contour of integration in the integral (3.1) for $G_+(\alpha)$ such that it passes along two straight lines: L_1 ($\arg\alpha = 5\pi/4$) and L_2 ($\arg\alpha = -\pi/4$) (Fig.1). We then close the path of integration with two arcs of large radius R: C_1 ($\alpha = Re^{i\theta}$, $\pi < \theta < 5\pi/4$) and C_2 ($\alpha = Re^{i\theta}$, $-\pi/4 < \theta < 0$) and an arc of small radius ρ in the neighbourhood of C_ρ (circumventing the branch point). It can be shown that

$$\int_{C_1+C_2} \frac{G(\zeta)}{\zeta - \alpha}\, d\zeta \to 0 \quad \text{as } R^{-2} \text{ when } R \to \infty$$

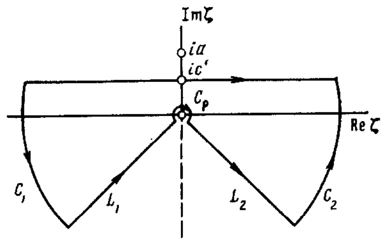

Fig.1

$$\int_{C_\rho} \frac{G(\zeta)}{\zeta-\alpha}\, d\zeta \to 0 \quad \text{as } \rho^{2/3} \text{ when } \rho \to 0$$

The integrand is analytic within the closed contour bounded by C_1, L_1, C_ρ, L_2 and C_2 and the initial line of integration and, consequently

$$G_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{L_1+L_2} \frac{g(z(\zeta))}{\zeta(\zeta-\alpha)(\zeta+i\delta)}\, d\zeta \tag{3.2}$$

Let us make the following change of variables

on L_2: $\zeta = re^{-i\pi/4}$, $g(z) = g(r^{4/3}) = K_{1/3}(^2/_3 r^3)/K_{2/3}(^2/_3 r^3)$
on L_1: $\zeta = re^{i5\pi/4}$, $g(z) = -g(r^{4/3})$

After transformations, the integral (3.2) can be reduced to the form

$$G_+(\alpha) = Q(\alpha), \quad Q(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{g(r^{4/3})\,[2t\delta + \sqrt{2}\,r(\delta - t)]}{S(t,r)\,R(r)}\, dr \tag{3.3}$$

$$S(t, r) = r^2 + \sqrt{2}rt + t^2, \quad R(r) = r^2 - \sqrt{2}\delta r + \delta^2$$

$$t = -i\alpha$$

The function $G_-(\alpha)$ is found in a similar way. By taking account of the fact that, on passing from integration along the line $\operatorname{Im}\zeta = d$ to integration along the lines L_1 and L_2, the integrand has a first-order pole in the domain bounded by the contour $C_1, L_1, C_\rho, L_2, C_2$ and the straight line $\operatorname{Im}\zeta = d_2$ we obtain

$$G_-(\alpha) = G(\alpha) - Q(\alpha) \tag{3.4}$$

4. Let us find the asymptotic form of the solution when $|x| \gg 1$, $x > 0$. We will obtain an expression for $G_+(\alpha)$ when $|\alpha| \ll 1$. Taking account of the expansion for the function $g(r^{4/3})$ for small r, we shall represent g in the following manner:

$$g(r^{4/3}) = \frac{r^{2/3}}{p}(1 - pr^{4/3} + O(r^{8/3})) = \tag{4.1}$$

$$\frac{r^{2/3}}{p}(1 - pr^{4/3}) + N_1(r) = \frac{r^{2/3}}{p} + N_2(r), \quad p = 3^{1/3}\frac{\Gamma(^2/_3)}{\Gamma(^1/_3)}$$

where $\Gamma(z)$ is the gamma function.

Substituting (4.1) into (3.3), we can represent $G_+(\alpha)$ in the form

$$(2\pi)^{1/2}G_+(\alpha) = I_1 + I_2 + I_3 + I_4$$

$$I_1 = \frac{1}{p}\int_0^\infty \frac{2t\delta(1 - pr^{4/3})}{r^{1/3}S(t,r)\,R(r)}\, dr, \quad I_2 = \int_0^\infty \frac{2t\delta N_1(r)}{S(t,r)\,R(r)}\, dr$$

$$I_3 = \frac{1}{p}\int_0^\infty \frac{\sqrt{2}(\delta - t)\,r}{r^{1/3}S(t,r)\,R(r)}\, dr, \quad I_4 = \int_0^\infty \frac{\sqrt{2}(\delta - t)\,rN_2(r)}{S(t,r)\,R(r)}\, dr$$

and make an estimation of the terms. By taking account of the representation $1/S(t, r) = r^{-2} - \sqrt{2}tr^{-3} + (t^2r^{-2} + \sqrt{2}t^3r^{-3})/S$, we get

$$I_1 = \frac{2\pi}{p\delta}\left(1 - \frac{1}{\sqrt{3}}\right)t^{-1/3} + \frac{4\pi}{p\delta^2\sqrt{3}}\,t^{2/3} + \frac{2t}{\delta}\ln t + O(t), \quad I_2 = O(t)$$

$$I_3 = \frac{2\pi}{p\delta\sqrt{3}}\,t^{-1/3} - \frac{2\pi}{p\delta^2}\left(1 + \frac{2}{\sqrt{3}}\right)t^{2/3} + C_1 + O(t), \quad I_4 = C_2 + O(t)$$

$$C_1 = \frac{1}{p}\int_0^\infty \frac{2 - \sqrt{2}\,r/\delta}{r^{1/3}R(r)}\, dr, \quad C_2 = \sqrt{2}\,\delta\int_0^\infty \frac{g(r^{4/3}) - r^{2/3}/p}{rR(r)}\, dr$$

Let us write the final expression as

$$G_+(\alpha) = \frac{t^{-1/3}}{p\delta\sqrt{2\pi}} + \frac{C_1 + C_2}{(2\pi)^{1/2}} - \frac{t^{2/3}}{p\delta^2\sqrt{2\pi}} + \frac{t\ln t}{\delta\pi\sqrt{2\pi}} + O(t) \tag{4.2}$$

For the function $f(x) \equiv f(x, 0)\ (x > 0)$, by using Theorem 41.1 from /7/, we get

$$f(x) = \frac{3^{1/6}\Gamma(1/3)}{2\pi\delta} x^{-1/3} + \frac{3^{-5/6}\Gamma(1/3)}{\pi\delta^2} x^{-5/3} + O(x^{-2}\ln x) \tag{4.3}$$

In doing this, the concentration on the surface is determined using formula (1.3).

5. Let us find the asymptotic form of the solution close to the point of discontinuity, that is, the solution when $|x| \ll 1$, $x \gtrless 0$. We initially obtain an expression for $G_+(\alpha)$ when $|\alpha| \gg 1$.

Let us transform integral (3.3) in the same way as was done in /8/, taking into account the expansion

$$g(r^{4/3}) = 1 - {}^1/_4 r^{-2} + O(r^{-4}), \quad r \gg 1$$

$$\frac{1}{S(t, r)} = \frac{1}{t^2} - \frac{\sqrt{2}\, r}{t^3} + \frac{T(t, r)}{t^3}, \quad T(t, r) = \frac{tr^2 + \sqrt{2}\, r^3}{S(t, r)}$$

$$\begin{aligned}(2\pi)^{3/2} G_+(\alpha) = {} & \frac{2\delta}{t} J_{-1} - \frac{\sqrt{2}\,\delta}{t^2} J_0 - \frac{\sqrt{2}}{t} J_0 + \frac{2\delta}{t^2}\int_0^\infty \frac{T(t, r)\, P(r)}{r}\, dr - \\ & \frac{2(\delta - t)}{t^3}\int_0^1 P(r)\, r\, dr - \frac{2(\delta - t)}{t^3}\int_1^\infty \frac{g(r^{4/3}) - 1}{R(r)}\, r\, dr + \\ & \frac{\sqrt{2}(\delta - t)}{t^3}\int_0^1 T(r)\, P(r)\, dr + \\ & \frac{\sqrt{2}(\delta - t)}{t^3}\int_1^\infty \frac{g(r^{4/3}) - 1}{R(r)}\, T(r, t)\, dr - \frac{2(\delta - t)}{t^3}\int_1^\infty \frac{r - {}^1/_2\sqrt{2}\, T(t, r)}{R(r)}\, dr\end{aligned} \tag{5.1}$$

$$P(r) = \frac{g(r^{4/3})}{R(r)}, \quad J_n = \int_0^\infty P(r)\, r^n\, dr$$

The fourth, seventh and eighth terms are of the order of t^{-3} while the fifth and sixth terms can be represented in the form of a sum of terms of the order of t^{-2} and t^{-3} respectively. The last term can be reduced to the form

$$\frac{1}{t^2}\left[\pi - 2\,\mathrm{arctg}\,\frac{\sqrt{2} - \delta}{\delta} - \ln(1 - \sqrt{2}\,\delta + \delta^2)\right] + 2\frac{\ln t}{t} - 4\delta\,\frac{\ln t}{t^3} + O(t^{-3})$$

We shall obtain the final expression for $G_+(\alpha)$ when $|\alpha| \gg 1$, by discarding the terms $O(t^{-3})$ in (5.1)

$$G_+(\alpha) = \frac{1}{(2\pi)^{3/2}}\left[\frac{b_1}{-i\alpha} + b_2\frac{\ln(-i\alpha)}{(i\alpha)^2} + \frac{b_3}{(i\alpha)^2} + b_4\frac{\ln(-i\alpha)}{(-i\alpha)^3}\right] + O(|\alpha|^{-3}) \tag{5.2}$$

$$\begin{aligned}& b_1 = 2\delta J_{-1} - \sqrt{2}\, J_0, \quad b_2 = 2, \quad b_3 = \int_0^1 P(r)\, r\, dr + 2\int_1^\infty \frac{g(r^{4/3}) - 1}{R(r)}\, dr - \\ & \sqrt{2}\,\delta J_0 + \pi - 2\,\mathrm{arctg}\,\frac{\sqrt{2} - \delta}{\delta} - \ln(1 - \sqrt{2}\,\delta + \delta^2), \quad b_4 = -4\delta\end{aligned}$$

Knowing the Fourier transformation of $G_+(\alpha)$ from $f(x)\ (x > 0)$ when $|\alpha| \gg 1$, we find **the original** $f(x)$ when $x \ll 1$, having made use of an analogue of Watson's lemma in the case when the function has a logarithmic singularity /4/:

$$f(x) = \frac{b_1}{2\pi} - \frac{1}{\pi} x \ln x + O(x^2 \ln x), \quad x > 0 \tag{5.3}$$

Let us pass to the limit as $\delta \to 0$ in this expression, that is, we shall find the solution corresponding to the boundary conditions (1.4).

By taking account of expansion (4.1) of the function $g(r^{4/3})$ for small r, we represent b_1 in the form of a sum of several terms in a similar way to the representation of $G_+(\alpha)$ in Sect.4. After evaluating the corresponding integrals and passing to the limit as $\delta \to 0$, we get

$$f(x) = a_1 - \pi^{-1} x \ln x, \quad |x| \ll 1, \quad x > 0 \tag{5.4}$$

$$a_1 = \frac{1}{\sqrt{2}\,\pi}\int_0^\infty \frac{r^{2/3}/p - g(r^{4/3})}{r^2}\, dr \approx 0.8822 \tag{5.5}$$

The asymptotic form of the solution when $x<0$, $|x|\ll 1$ are determined in a similar way and corresponds to (5.4) when $\ln x$ is replaced by $\ln(-x)$.

The concentration on the surface is determined using the formula

$$c/c_0 = 1 - k\,(0.8822 - \pi^{-1}x\ln|x|),\quad |x|\ll 1,\quad k\ll 1 \tag{5.6}$$

Let us now compare this with the solution obtained from the theory of a diffusing boundary layer. In the boundary-layer approximation, ignoring longitudinal diffusion, we shall write the initial problem in the form $(x>0)$

$$y\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y^2};\quad y=0:\ \frac{\partial f}{\partial y} = -1;\quad y\to\infty:\ f\to 0$$

This problem has the solution

$$f(x,y) = -y + 3^{-2/3}\Gamma(2/3)\left[\exp\left(-\frac{y^3}{9x}\right)x^{1/3} + \frac{y}{3}\int_0^{y/x^{1/3}} \xi\exp\left(-\frac{\xi^3}{9}\right)d\xi\right]$$

$$f(x,0) = 3^{-2/3}\Gamma(2/3)\,x^{1/3},\quad \frac{\partial f}{\partial x}(x,0) = 3^{-5/3}\Gamma(2/3)\,x^{-2/3}$$

It is seen that both the true value (taking **longitudinal** diffusion into account) of the required function on the surface when $x=0$: $f\approx 0.8822$ differs from the value given by boundary-layer theory, $f=0$, and the true character of its behaviour when $x\to+0$: $\partial f/\partial x\sim|\ln x|$ differs from the dependence specified by the boundary-layer solution: $\partial f/\partial x\sim x^{-2/3}$.

A comparison of the value of the concentration on the surface when $x=0$ with the solution obtained by a numerical method /1/ shows that the difference between the results does not exceed 5%.

6. It now remains to verify assumption (1.8) made in Sect.1 concerning the nature of the decay of the solution when $x\to-\infty$. It is sufficient to carry out this verification when $y=0$.

The Fourier transform from $f(x)$ when $x<0$ can be writtten in the form

$$G_-(\alpha) = -\frac{1}{(2\pi)^{3/2}}\int_{id-\infty}^{id+\infty} G^*(\zeta)\,d\zeta \tag{6.1}$$

$$G^*(\zeta) = \frac{K_{1/3}\,(2/3\zeta^3 e^{i\pi/2})}{\zeta(\zeta+i\delta)(\zeta-\alpha)\,K_{2/3}\,(2/3\zeta^3 e^{i\pi/2})},\quad \operatorname{Im}\alpha<d<a$$

Let us now augment the contour of integration in the upper half plane with a semicircular arc C_R of large radius R: $|\zeta - id| = R$. It can be shown that

$$\int_{C_R} G^*\,d\zeta \to 0 \ \text{ as } R^{-2} \text{ when } R\to\infty$$

The function $G^*(\zeta)$ is meromorphic in the half plane $\operatorname{Im}\zeta\geqslant d$ and therefore

$$\int_{id-\infty}^{id+\infty} G^*(\zeta)\,d\zeta = 2\pi i\sum_{k=1}^{\infty}\operatorname{Res}\,[G^*(\zeta,\zeta_k)] \tag{6.2}$$

Here, ζ_k are the simple poles of the function $G^*(\zeta)$ or the **zeros** of the function $K_{2/3}$ $(2/3\zeta^3 e^{i\pi/2})$ so that the residues of the function G^* are finite at these points.

From a relationship which follows from the rules for the differentiation of Bessel functions

$$\eta K_{2/3}'(\eta) + 2/3\,K_{2/3}(\eta) = -\eta\,K_{1/3}(\eta)$$

it follows that

$$\operatorname{Res}\,[G^*(\zeta,\zeta_k)] = K_{1/3}(\eta_k)\left[\zeta_k(\zeta_k+i\delta)(\zeta_k-\alpha)\,K_{2/3}'(\eta_k)\,\frac{d\eta}{d\zeta}(\zeta_k)\right]^{-1} = 3i\,[4\zeta_k^2(\zeta_k+i\delta)(\zeta_k-\alpha)]^{-1} \tag{6.3}$$

On the basis of (6.1)-(6.3), we get

$$G_-(\alpha) = \frac{3}{4\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{1}{s_k^2(s_k+\delta)(s_k+i\alpha)},\quad \zeta_k = is_k,\quad s_k = \sqrt[3]{3/2\,t_k}$$

where $-it_k$ are the zeros of the Bessel function $K_{2/3}(\eta)$, $t_k>0$ are real numbers and, moreover, $t_1<t_2<t_3<\ldots$.

When $x<0$, the required function $f(x)$ is determined in the following manner:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{i\tau'-\infty}^{i\tau'+\infty} G_-(\alpha)\,e^{-i\alpha x}\,d\alpha,\quad 0<\tau'<a$$

The function $G_-(\alpha)$ satisfies the conditions of Jordan's lemma in the upper half plane and has an infinite number of simple poles on the positive imaginary axis at the points $\alpha_k = is_k$. Then,

$$f(x)=\frac{3}{4}\sum_{k=1}^{\infty}\frac{e^{s_k x}}{s_k^3(s_k+\delta)}<\frac{3}{4}\frac{1}{s_1^3}\sum_{k=1}^{\infty}e^{s_k x}$$

whence, as $x\to-\infty$

$$|f|<Ae^{(s_1-\varepsilon)x},\quad A\geqslant\frac{3}{4s_1^3}$$

where ε is an arbitrary small finite positive quantity.

So, it is possible to put a in Eq.(1.8) equal to any positive number which is strictly less than s_1 ($s_1\approx 1.014$).

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